Dept. of Mathematics, St. Xavier's College, Palayamkottai

Vector Spaces

Unit-I

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Content

- Any entity involving both magnitude and direction is called a "vector."

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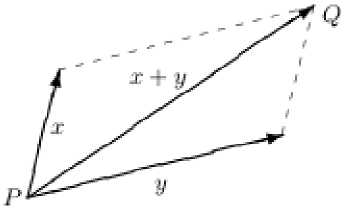
- A vector is represented by an arrow whose length denotes the magnitude of the vector and whose direction represents the direction of the vector.

Parallelogram Law for Vector Addition.

The sum of two vectors x and y that act at the same point is the vector beginning at that is represented by the diagonal of parallelogram having x and y as adjacent sides.



Besides the operation of vector addition, there is another natural operation that can be performed on vectors–the length of a vector may be magnified or contracted. This operation, called scalar multiplication, consists of multiplying the vector by a real number.

Problems

1. Determine whether the vectors emanating from the origin and terminating at the following pairs of points are parallel.

(a) $(3, 1, 2)$ and $(6, 4, 2)$

(b) $(-3, 1, 7)$ and $(9, -3, -21)$

(c) $(5, -6, 7)$ and $(-5, 6, -7)$

(d) $(2, 0, -5)$ and $(5, 0, -2)$

2. Find the equations of the lines through the following pairs of points in space.

(a) $(3, -2, 4)$ and $(-5, 7, 1)$

(b) $(2, 4, 0)$ and $(-3, -6, 0)$

(c) $(3, 7, 2)$ and $(3, 7, -8)$

(d) $(-2, -1, 5)$ and $(3, 9, 7)$

Definition- Vector space

Definition

A vector space (or linear space) V over a field F consists of a set on which two operations (called addition and scalar multiplication, respectively) are defined so that for each pair of elements x, y in V there is a unique element $x + y$ in V, and for each element a in F and each element x in V there is a unique element ax in V, such that the following conditions hold.

Definition-Vector space

Definition

1. For all $x, y \in V, x + y = y + x$.
2. For all $x, y, z \in V, (x + y) + z = x + (y + z)$.
3. There exist an element in V denoted by 0 such that $x + 0 = x$ for each $x \in V$.
4. For each element $x \in V$ there exists an element y in V such that $x + y = 0$.
5. For each element $x \in V$, $1x = x$.
6. For each pair of elements a, b in F and each element $x \in V$, $(ab)x = a(bx)$.
7. For each element $a \in F$ and each pair of elements $x, y \in V$, $a(x + y) = ax + ay$.
8. For each pair of elements $a, b \in F$ and each elements $x \in V$, $(a + b)x = ax + bx$.

NOTE:

1. The elements $x + y$ and ax are called the sum of x and y and the product of a and x, respectively.
2. The elements of the field F are called scalars and the elements of the vector space V are called vectors.
3. The word “vector” is now being used to describe any element of a vector space.

Examples

1. Let F be a field. The set of all $n-$tuples $F^n = \{(a_1, a_2, \ldots, a_n) | a_i \in F, i = 1, 2, \ldots, n\}$ is a vector space over F with coordinatewise addition and scalar multiplication.

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2. Let $M_{m\times n}(F)$ denotes the set of all $m \times n$ matrices with entries from the field F. Then $M_{m\times n}(F)$ is a vector space over F with matrix addition and scalar multiplication.

Examples

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2. Let $M_{m \times n}(F)$ denotes the set of all $m \times n$ matrices with entries from the field F. Then $M_{m \times n}(F)$ is a vector space over F with matrix addition and scalar multiplication.
3. Let S be any nonempty set and F be any field, and let $F(S, F)$ denote the set of all functions from S to F . Two functions f and g in $F(S, F)$ are called equal if $f(s) = g(s)$ for each $s \in S$. The set $F(S, F)$ is a vector space with the operations of addition and scalar multiplication defined for $f, g \in F(S, F)$ and $c \in F$ by

$$(f + g)(s) = f(s) + g(s) \text{ and } (cf)(s) = c[f(s)]$$

Examples

4. The set of all polynomials with coefficients from a field F is a vector space over F with usual addition and scalar multiplication of polynomials.

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4. The set of all polynomials with coefficients from a field F is a vector space over F with usual addition and scalar multiplication of polynomials.
5. Let V consist of all sequences $\{a_n\}$ in F that have only a finite number of nonzero terms a_n. If $\{a_n\}$ and $\{b_n\}$ are in V and $t \in F$, define $\{a_n\} + \{b_n\} = \{a_n + b_n\}$ and $t\{a_n\} = \{ta_n\}$.

Cancellation Law for Vector Addition

Theorem (Cancellation Law for Vector Addition)

If x, y and z are vectors in a vector space V such that $x + z = y + z$, then $x = y$.

Cancellation Law for Vector Addition

Proof.

As $z \in V$ and V is a vector space, there exists a vector $v \in V$ such that $z + v = 0$. Hence

$$\begin{aligned} x &= x + 0 \\ &= x + (z + v) \\ &= (x + z) + v \\ &= (y + z) + v \\ &= y + (z + v) \\ &= y + 0 \\ &= y \end{aligned}$$

$\square$

Corollary

The vector 0 in any vector space is unique.

Proof.

Suppose $0, 0' \in V$ such that $x + 0 = x$ and $x + 0' = x$. Thus $x + 0 = x + 0'$. Then by cancellation law for vector addition $0 = 0'$. Hence proved. □

Note:
The vector 0 is called the zero vector of V.

Corollary

The vector y described in condition (4) of the definition of Vector space is unique.

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Proof.

!!!!!! Just try by yourself !!!!!! $\square$

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Proof.

!!!!!! Just try by yourself !!!!!! $\square$

Note:

the vector y is called the additive inverse of x and is denoted by $-x$.

Theorem

In any vector space V, the following statements are true:

(a) $0x = 0$ *for each* $x \in V$.

(b) $(-a)x = -(ax) = a(-x)$ *for each* $a \in F$ *and each* $x \in V$.

(c) $a0 = 0$ *for each* $a \in F$.

Proof.

(a) Let x be any vector in V. Then

$$\begin{aligned} 0x + 0x &= (0+0)x \\ &= 0x \\ &= 0x + 0 \\ &= 0 + 0x \end{aligned}$$

By Cancellation Law for Vector Addition, we have $0x = 0$.
(b) Since $-(ax)$ is the unique additive inverse of ax,

$$ax + [-(ax)] = 0. \tag{1}$$

Now,

$$ax + (-a)x = [a + (-a)]x = 0x = 0 \tag{2}$$

From equation (1) and (2), we have $(-a)x = -(ax)$. In particular, $(-1)x = -x$. Now

$$a(-x) = a[(-1)x] = [a(-1)]x = (-a)x.$$

The proof of (c) is similar to (a) $\square$

Definition

A subset W of a vector space V over a field F is called a subspace of V if W is a vector space over F with the operations of addition and scalar multiplication defined on V.

Note:
In any vector space V, V and $\{0\}$ are subspaces.

Theorem

Let V be a vector space and W a subset of V. Then W is a subspace of V if and only if the following three conditions hold for the operations defined in V.

1. $0 \in W$
2. $x + y \in W$ *whenever* $x \in W$ *and* $y \in W$.
3. $cx \in W$ *whenever* $c \in F$ *and* $x \in W$.

Remarks:

The transpose A^t of an $m \times n$ matrix A is the $n \times m$ matrix obtained from A by interchanging the rows with the columns; that is, $(A^t)_{ij} = A_{ji}$. A symmetric matrix is a matrix A such that $A^t = A$.

1. The set W of all symmetric matrices in $M_{n\times n}(F)$ is a subspace of $M_{n\times n}(F)$.

Theorem

Any intersection of subspaces of a vector space V is a subspace of V.

Remark:

1. Union of subspaces need not be a subspace
2. Union of two subspaces of a vector space V is a subspace of V if and only if one of the subspaces contains other.

Definition

Let V be a vector space and S a nonempty subset of V. A vector $v \in V$ is called a linear combination of vectors of S if there exist a finite number of vectors $u_1, u_2, \dots, u_n$ in S and scalars $a_1, a_2, \dots, a_n$ in F such that $v = a_1u_1 + a_2u_2 + \dots + a_nu_n$. In this case we also say that v is a linear combination of $u_1, u_2, \dots, u_n$ and call $a_1, a_2, \dots, a_n$ the coefficients of the linear combination.

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Problem

Express $(2, 6, 8)$ *as a linear combination of* $u_1 = (1, 2, 1), u_2 = (-2, -4, -2), u_3 = (0, 2, 3),$

Problem

Prove that $2x^3 - 2x^2 + 12x - 6$ *is a linear combination of* $x^3 - 2x^2 - 5x - 3$ *and* $3x^3 - 5x^2 - 4x - 9$. *Also show that* $3x^3 - 2x^2 + 7x + 8$ *is not a linear combination of* $x^3 - 2x^2 - 5x - 3$ *and* $3x^3 - 5x^2 - 4x - 9$.

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Definition

Let S be a nonempty subset of a vector space V. The span of S, denoted span(S), is the set consisting of all linear combinations of the vectors in S. For convenience, we define span($\emptyset$) = $\{0\}$.

Example

In $\mathbb{R}^3$, the span of the set $\{(1,0,0),(0,1,0)\}$ consists of all vectors in $\mathbb{R}^3$ that have the form $a(1,0,0)+b(0,1,0)=(a,b,0)$ for some scalars a and b. Thus the span of $\{(1,0,0),(0,1,0)\}$ contains all the points in the xy-plane.

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Theorem

The span of any subset S of a vector space V is a subspace of V. Moreover, any subspace of V that contains S must also contain the span of S.

Definition

A subset S of a vector space V generates (or spans) V if span(S) = V. In this case, we also say that the vectors of S generate (or span) V.

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Problem

Show that the vectors $(1, 1, 0), (1, 0, 1)$, *and* $(0, 1, 1)$ *generate* $\mathbb{R}^3$

Problem

Show that the polynomials $x^2 + 3x - 2, 2x^2 + 5x - 3$, *and* $-x^2 - 4x + 4$ *generate* $P_2(\mathbb{R})$

Problem

Show that the matrices $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ *and* $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ *generate* $M_{2\times 2}(\mathbb{R})$.

Linear Dependence and Linear Independence

Definition

A subset S of a vector space V is called linearly dependent if there exist a finite number of distinct vectors $u_1, u_2, \ldots, u_n$ in S and scalars $a_1, a_2, \ldots, a_n$, not all zero, such that $a_1u_1 + a_2u_2 + \ldots + a_nu_n = 0$.

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Problem

Let $S = \{(1, 3, -4, 2), (2, 2, -4, 0), (1, -3, 2, -4), (-1, 0, 1, 0)\}$ *be a set in* $\mathbb{R}^4$. *Show that* S *is linearly dependent in* $\mathbb{R}^4$

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Problem

Let $S = \{(1, 3, -4, 2), (2, 2, -4, 0), (1, -3, 2, -4), (-1, 0, 1, 0)\}$ be a set in $\mathbb{R}^4$. Show that S is linearly dependent in $\mathbb{R}^4$

Problem

Show that the set
$$\left\{ \begin{bmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{bmatrix}, \begin{bmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{bmatrix}, \begin{bmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{bmatrix} \right\} \text{ in } M_{2\times 3}(\mathbb{R})$$
is linearly dependent.

Definition

A subset S of a vector space that is not linearly dependent is called linearly independent.

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Problem

Show that the set $S = \{(1,0,0,-1),(0,1,0,-1),(0,0,1,-1),(0,0,0,1)\}$ *is linearly independent*

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Show that the set $S = \{(1, 0, 0, -1), (0, 1, 0, -1), (0, 0, 1, -1), (0, 0, 0, 1)\}$ *is linearly independent*

Problem

Let $p_k(x) = x^k + x^{k+1} + \cdots + x^n, k = 0, 1, \ldots, n$. *Show that the set* $\{p_0(x), p_1(x), \ldots, p_n(x)\}$ *is linearly independent in* $P_n(F)$

Theorem

Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly dependent, then S_2 is linearly dependent.

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Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_2 is linearly independent, then S_1 is linearly independent.

Theorem

Let S be a linearly independent subset of a vector space V, and let v be a vector in V that is not in S. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in span(S)$.

Note:
The empty set is linearly independent, for linearly dependent sets must be nonempty.

Definition

A basis β for a vector space V is a linearly independent subset of V that generates V.

Example

We know that $span(\phi) = \{0\}$ and ϕ is linearly independent, hence ϕ is a basis for the zero vector space.

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Example

In F^n , let $e_1 = (1, 0, 0, ..., 0), e_2 = (0, 1, 0, ..., 0), ..., e_n = (0, 0, ..., 0, 1)$; It is easy to see that $\{e_1, e_2, ..., e_n\}$ is a basis for F^n and is called the standard basis for F^n .

Example

In $M_{m\times n}(F)$, let E_{ij} denote the matrix whose only nonzero entry is a 1 in the ith row and jth column. Then $\{E_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ is a basis for $M_{m\times n}(F)$.

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Example

In $P_n(F)$ the set $\{1, x, x^2, \ldots, x^n\}$ is a basis. This basis is called the standard basis for $P_n(F)$

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Example

The set $\{1, x, x^2, \ldots\}$ is a basis for $P(F)$.

Theorem

Let V be a vector space and $\beta = \{u_1, u_2, \ldots, u_n\}$ be a subset of V. Then β is a basis for V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of β, that is, can be expressed in the form

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$$

for unique scalars $a_1, a_2, \ldots, a_n$.

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$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$$

for unique scalars $a_1, a_2, \ldots, a_n$.

Theorem

If a vector space V is generated by a finite set S, then some subset of S is a basis for V. Hence V has a finite basis.

Problem

Let $S = \{(2, -3, 5), (8, -12, 20), (1, 0, -2), (0, 2, -1), (7, 2, 0)\}$ *be subset of* $\mathbb{R}^3$*. Extract a basis for* $\mathbb{R}^3$ *which is a subset of* S*.*

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Theorem (Replacement Theorem)

Let V *be a vector space that is generated by a set* G *containing exactly* n *vectors, and let* L *be a linearly independent subset of* V *containing exactly* m *vectors. Then* $m \leq n$ *and there exists a subset* H *of* G *containing exactly* $n - m$ *vectors such that* $L \cup H$ *generates* V.

Problem

Let $S = \{(2,-3,5),(8,-12,20),(1,0,-2),(0,2,-1),(7,2,0)\}$ be subset of $\mathbb{R}^3$. Extract a basis for $\mathbb{R}^3$ which is a subset of S.

Theorem (Replacement Theorem)

Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V containing exactly m vectors. Then $m \leq n$ and there exists a subset H of G containing exactly $n-m$ vectors such that $L \cup H$ generates V.

Corollary

Let V be a vector space having a finite basis. Then every basis for V contains the same number of vectors.

Bases and Dimension

Definition

A vector space is called finite-dimensional if it has a basis consisting of a finite number of vectors. The unique number of vectors in each basis for V is called the dimension of V and is denoted by $dim(V)$. A vector space that is not finite-dimensional is called infinite-dimensional.

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Examples

1. $\dim(\{0\}) = 0$
2. $\dim(\mathbb{C}) = 1$ over the field $\mathbb{C}$
3. $\dim(\mathbb{C}) = 2$ over the field $\mathbb{R}$
4. $\dim(F^n) = n$
5. $\dim(M_{m\times n}) = mn$
6. $\dim(P_n(F)) = \cancel{n}(?????)$

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Definition

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4. $\dim(F^n) = n$
5. $\dim(M_{m\times n}) = mn$
6. $\dim(P_n(F)) = \cancel{n}(?????) = n + 1$

Theorem

Let V be a vector space with dimension n.
(a) Any finite generating set for V contains at least n vectors, and a generating set for V that contains exactly n vectors is a basis for V.
(b) Any linearly independent subset of V that contains exactly n vectors is a basis for V.
(c) Every linearly independent subset of V can be extended to a basis for V.

1. Show that $\{x^2 + 3x - 2, 2x^2 + 5x - 3, -x^2 - 4x + 4\}$ is a basis for $P_2(\mathbb{R}$.
2. Show that $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ is a basis for $M_{2\times 2}(\mathbb{R})$
3. Show that the set $S = \{(1, 0, 0, -1), (0, 1, 0, -1), (0, 0, 1, -1), (0, 0, 0, 1)\}$ is a basis for $\mathbb{R}^4$
4. Let $p_k(x) = x^k + x^{k+1} + \cdots + x^n, k = 0, 1, \ldots, n$. Show that the set $\{p_0(x), p_1(x), \ldots, p_n(x)\}$ is a basis for $P_n(F)$

Theorem

Let W be a subspace of a finite-dimensional vector space V. Then W is finite-dimensional and $\dim(W) \leq \dim(V)$. Moreover, if $\dim(W) = \dim(V)$, then $V = W$.

1. Let $W = \{(a_1, a_2, a_3, a_4, a_5) \in F^5 : a_1 + a_3 + a_5 = 0, a_2 = a_4\}$. Is W a subspace of F^5? If so, find its basis and dimension.
2. Is the set of all $n \times n$ diagonal matrices a subspace of $M_{n \times n}$. If so, find its basis and dimension.
3. Is the set of all $n \times n$ symmetric matrices a subspace of $M_{n \times n}$. If so, find its basis and dimension.

Corollary

If W is a subspace of a finite-dimensional vector space V, then any basis for W can be extended to a basis for V.

Lagrange polynomials

Let $c_0, c_1, ..., c_n$ be distinct scalars in an infinite field F. The polynomials $f_0(x), f_1(x), ..., f_n(x)$ defined by

$$\begin{aligned} f_i(x) &= \frac{(x-c_1)(x-c_2)\cdots(x-c_{i-1})(x-c_{i+1})\dots(x-c_n)}{(c_i-c_1)(c_i-c_2)\cdots(c_i-c_{i-1})(c_i-c_{i+1})\dots(c_i-c_n)} \\ &= \Pi_{k=0,k\neq i}^{n} \frac{x-c_k}{c_i-c_k} \end{aligned}$$

are called the Lagrange polynomials (associated with $c_0, c_1, ..., c_n$).

Linear Transformations

Definition

Let V and W be vector spaces (over F). We call a function $T : V \to W$ a linear transformation from V to W if, for all $x, y \in V$ and $c \in F$, we have

1. $T(x + y) = T(x) + T(y)$
2. $T(cx) = cT(x)$

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1. $T(x + y) = T(x) + T(y)$
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Example

Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a_1, a_2) = (2a_1 + a_2, a_1)$. Then T is a linear Transformation.

Problem

Prove that every linear transformation T from $\mathbb{R}$ to $\mathbb{R}$ is of the form $T(x) = cx$ for some fixed $c \in \mathbb{R}$

Problem

Prove that every linear transformation T *from* $\mathbb{R}$ *to* $\mathbb{R}$ *is of the form* $T(x) = cx$ *for some fixed* $c \in \mathbb{R}$

Problem

Prove that every linear transformation T *from* $\mathbb{R}^2$ *to* $\mathbb{R}^2$ *is of the form* $T(x, y) = (ax + by, cx + dy)$ *for some fixed* $a, b, c, d \in \mathbb{R}$

Thank you